Topology and its Applications 17 (1984) 173–187
North-Holland

EMBEDDING E^n/G IN EUCLIDEAN SPACE

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Received 22 November 1982
Revised 11 April 1983

It is shown that if G is an arbitrary upper semicontinuous decomposition of E^n for which $\pi(N_G)$ embeds in S^m for some $m \geq 3$, then the decomposition space E^n/G embeds as a closed subset of E^{n+m+1}. The proof consists of constructing a cell-like upper semicontinuous decomposition $\tilde{G}$ of E^{n+m+1}which intersects E^n to yield precisely G and using Edwards' Cell-Like Approximation Theorem to show that $\tilde{G}$ is shrinkable. As an immediate corollary, E^n/G embeds in E^{n+2k+2} whenever G is an arbitrary k-dimensional upper semicontinuous decomposition of E^n. This is an improvement of $(n-1)$-dimensions over the corresponding dimension theoretic result and examples due to Daverman show that this result is sharp in case n is odd and off by no more than one dimension in case n is even.

AMS Subj. Class. (1980): Primary, 54B15; Secondary, 54C25

upper semicontinuous decomposition embedding
finite dimensional decomposition

0. Introduction

If G is a k-dimensional upper semicontinuous decomposition of an n-dimensional separable metric space X, then $\dim X/G \leq n+k+1$ and basic dimension theory implies that X/G embeds in Euclidean $(2n+2k+3)$-space. If $X = E^n$ we can get a better restriction on the dimension of the decomposition space, namely $\dim E^n/G \leq n+k$ and E^n/G embeds in Euclidean $(2n+2k+1)$-space. It is natural to ask whether this is the best possible result. Specifically, we pose the following question.

For each $k \geq 0$ and $n \geq 1$, what is the smallest dimensional Euclidean space that admits an embedding of every decomposition space arising from a k-dimensional upper semicontinuous decomposition G of E^n?

It is well known that if G is assumed to be cell-like and $n \geq 4$, then $k \leq n$ and the decomposition space embeds in E^{n+1} [1, Chapter 26]. The existence of non-shrinkable cell-like decompositions of E^n with exactly one nondegenerate element shows that this is best possible. Thus, this paper specifically addresses the question in case G is not cell-like.

0166-8641/84/$3.00

Section 1 contains some definitions, notation, and basic concepts used throughout the paper. In Section 2, we construct from a given finite dimensional (non-cell-like) upper semicontinuous decomposition G of E^n and a given embedding ψ of $\pi(N_G)$ (defined in Section 1) into some m-sphere S^m a cell-like upper semicontinuous decomposition $\tilde{G}$ of E^{n+m+1} whose restriction to E^n yields the original decomposition G. The main results are stated in Section 3. In particular, we state that $\tilde{G}$ is shrinkable in case $m \geq 3$ and this implies that E^n/G embeds in E^{n+m+1}. This provides a partial answer to the question posed above, namely that E^n/G embeds in E^{n+2k+2} in case G is k-dimensional ($k \geq 1$), an improvement of $n-1$ dimensions over the dimension theoretic result. These techniques do not work in case $k=0$ and we include an appendix at the end of the paper which contains a proof that E^n/G embeds in E^{n+2} in case G is 0-dimensional. The shrinkability of $\tilde{G}$ is proved in Section 4 and we construct some examples and discuss the sharpness of the results in Section 5.

The author expresses his appreciation to Professors R.J. Daverman and J.J. Walsh for many enlightening conversations and for their helpful advice.

1. Definitions, notation, and basic concepts

An upper semicontinuous decomposition G of a space X is said to be (*closed*) *k-dimensional* provided (the closure of) $\pi(N_G)$ is k-dimensional. Here $\pi: X \to X/G$ denotes the decomposition map and N_G, the *nondegeneracy set of* G, denotes the union of the nondegenerate elements of G. We use the abbreviation usc for *upper semicontinuous* and we use H_G to denote the *collection of nondegenerate elements of* G. The basic theory of usc decompositions of manifolds appears in [1].

Let G be an usc decomposition of Euclidean n-space E^n. An open cover $\mathcal{U}$ of E^n is G-saturated if whenever $g \in G$, $U \in \mathcal{U}$, and $g \cap U \neq \emptyset$, then $g \subset U$. The decomposition G is *shrinkable* provided for each G-saturated open cover $\mathcal{U}$ of E^n and for each open cover $\mathcal{V}$ of E^n, there is a homeomorphism h of E^n such that for each $x \in E^n$, $\{x, h(x)\}$ is contained in some element of $\mathcal{U}$, and for each $g \in G$, $h(g)$ is contained in some element of $\mathcal{V}$. One consequence of *Bing's Shrinkability Criterion* is that if G is shrinkable, then E^n/G is homeomorphic to E^n [1, Chapter 5].

A metric space X satisfies the *Disjoint Disks Property*, abbreviated as DDP, in case arbitrary pairs of maps of the 2-disk B^2 into X can be approximated by maps with disjoint images. This means that given maps $f_1, f_2: B^2 \to X$ and a positive number ε, there are maps $\mu_1, \mu_2: B^2 \to X$ with f_i ε-close to μ_i for $i \in \{1, 2\}$ and $\mu_1(B^2) \cap \mu_2(B^2) = \emptyset$. A decomposition G is *cell-like* if each $g \in G$ is *cell-like*, that is, each g is null-homotopic in every neighborhood of itself. The principal method that we use in detecting that a cell-like usc decomposition of E^n is shrinkable appears in the *Cell-like Approximation Theorem* of R.D. Edwards.

Cell-Like Approximation Theorem. [2] and [1, Chapter 24]. *Let G be a cell-like* usc *decomposition of an n-manifold M where $n \geq 5$. Then G is shrinkable if and only if M/G is finite dimensional and satisfies the* DDP.

Given an usc decomposition G of E^n and a positive number ε, $G(\varepsilon)$ is the decomposition of E^n whose nondegenerate elements are the elements of G with diameter greater than or equal to ε. It is easy to see that if G is usc, then $G(\varepsilon)$ is usc and $N_{G(\varepsilon)}$ is a closed subset of E^n.

Throughout this paper, we make no distinction between E^n and $E^n \times \{0\} \subset E^n \times E^m = E^{n+m}$ nor between S^m and $\{0\} \times S^m \subset E^n \times E^{m+1} = E^{n+m+1}$. S^m always denotes the unit m-sphere in E^{m+1}. We use the usual normed linear structure on E^n: for elements x and y in E^n, $|x|$ is the usual Euclidean norm on x and $|x-y|$ is the usual Euclidean distance between x and y. For subsets $A \subset E^n$ and $B \subset E^m$, $A+B=\{a+b \in E^{n+m} \mid a \in A \text{ and } b \in B\}$. For a compact subset g of E^n and $\varepsilon > 0$, $\delta(g)$ is the diameter of g, $\operatorname{co}(g)$ is the convex hull of g, and $B_\varepsilon(g, E^n)$ is the ε-neighborhood of g in E^n. cl U denotes the closure of the subset U of E^n.

2. The decomposition $\tilde{G}$

Let G be a finite dimensional usc decomposition of E^n and let $\psi: \pi(N_G) \to S^m = \{0\} \times S^m \subset E^n \times E^{m+1}$ be an embedding of the image of the nondegeneracy set of G under the decomposition map into the m-sphere S^m. It is the purpose of this section to construct a finite dimensional cell-like usc decomposition $\tilde{G}$ of E^{n+m+1} with $\tilde{G} \cap E^n = G$, where $\tilde{G} \cap E^n = \{g \cap E^n \mid g \in \tilde{G}\}$. In Section 3 we show that for $m \geq 3$ the decomposition $\tilde{G}$ is shrinkable. This induces an embedding of E^n/G into E^{n+m+1} and in particular shows that if G is a k-dimensional ($k \geq 1$) usc decomposition of E^n, then E^n/G embeds in E^{n+2k+2}. We begin the construction of $\tilde{G}$ by introducing first an intermediate decomposition $\bar{G}$ of E^{n+m+1}.

For $g \in H_G$, let $a_g = \psi(\pi(g)) \in S^m$ and let A_g be the straight line segment in E^{m+1} of length $\delta(g)$ from 0 to $\delta(g) \cdot a_g$. Define $\hat{g}$ by $\hat{g} = (g + A_g) \cup (\operatorname{co}(g) + \delta(g) a_g) = \{x + t a_g \mid x \in g \text{ and } 0 \leq t \leq \delta(g), \text{ or } x \in \operatorname{co}(g) \text{ and } t = \delta(g)\}$. $\hat{g}$ is the product of g with an interval of length $\delta(g)$ in the direction determined by $\psi(\pi(g))$ 'capped off' at the $\delta(g)$ level with a copy of the convex hull of g. Notice that $\hat{g}$ is cell-like and $\hat{g} \cap E^n = g$. We would like to be able to say that the collection $\hat{G} = \{\hat{g} \mid g \in H_G\}$ determines an usc decomposition of E^{n+m+1} by trivially extending $\hat{G}$ to E^{n+m+1} and then hope that this decomposition is shrinkable. However, this is not the case. It is the 'capping off' of $\hat{g}$ with a copy of $\operatorname{co}(g)$ in order to gain cell-likeness that causes us to lose upper semicontinuity. We remedy this by first giving up cell-likeness in order to obtain upper semicontinuity, and later on adding in cell-likeness.

Let E_g denote the upper 'half space' of E^{n+m+1} spanned by E^n and a_g, that is, $E_g = \{x + t a_g \mid x \in E^n, t \geq 0\}$, and let $\mathring{E}_g = E_g \backslash E^n$. We are ready to define the decomposition $\bar{G}$. For $g \in H_G$, let $\bar{g} = g \cup \{y \in \mathring{E}_g \mid \text{every neighborhood of } y \text{ in } E^{n+m+1}$ intersects some element of $\hat{G}$ nontrivially}. In particular, notice that $\hat{g} \subset \bar{g}$ and $\bar{g} \cap E^n = g$. Let $\bar{G}$ be the decomposition of E^{n+m+1} with $H_{\bar{G}} = \{\bar{g} \mid g \in H_G\}$. We suspect that $\bar{G}$ is usc since $\bar{g}$ is constructed so that if $h \in G$ and h is very 'close' to g, then $\hat{h}$ (and thus hopefully $\bar{h}$) is 'close' to $\bar{g}$. We confirm this suspicion in Propositions 2.1 and 2.2 below.

Proposition 2.1. (A) $\bar{g} \subset \text{co}(g) + A_g = \{x + ta_g \mid x \in \text{co}(g), 0 \leq t \leq \delta(g)\}$. (B) *Suppose $x \in E^n$ and there is an $\varepsilon > 0$ so that $g \cap \text{cl } B_\varepsilon(x, E^n) = \emptyset$. Then $\bar{g} \cap B_\varepsilon(x, E^{n+m+1}) = \emptyset$.* (C) *$\bar{g}$ is closed in E_g and therefore in E^{n+m+1}.*

Note that Proposition 2.1(A) and (C) immediately implies that $\bar{g}$ is compact.

Proof. (A) Let $y \in \bar{g} \cap \mathring{E}_g$. Then $y = x + ta_g$ where $x \in E^n$ and $t > 0$. By the definition of $\bar{g}$, we can find a sequence $z_i = x_i + t_i a_{g_i} \in \hat{g}_i \in \hat{G}$ where $x_i \in \text{co}(g_i)$ and $0 \leq t_i \leq \delta(g_i)$ so that $z_i \to y$. But then $x_i \to x$, $t_i \to t$, and since $t \neq 0$, $a_{g_i} \to a_g$. Applying ψ^{-1} to the sequence a_{g_i} gives $\pi(g_i) \to \pi(g)$ so that, in particular, limsup $\delta(g_i) \leq \delta(g)$, in which case $0 \leq t \leq \delta(g)$. To complete the proof, observe that since $x_i \in \text{co}(g_i)$, $x_i \to x$, and $\pi(g_i) \to \pi(g)$, then $x \in \text{co}(g)$ and $y = x + ta_g \in \text{co}(g) + A_g$.

(B) Suppose that $\bar{g} \cap B_\varepsilon(x, E^{n+m+1}) \neq \emptyset$. Choose $y \in \bar{g}$ with $|x - y| < \varepsilon$ and $z_i \in \hat{g}_i \in \hat{G}$ so that $z_i \to y$. Since $\bar{g} \cap E^n = g$ and $g \cap B_\varepsilon(x, E^n) = \emptyset$, $y \notin E^n$ and therefore $\pi(g_i) \to \pi(g)$ (as shown above) and we may choose an integer m so large that $g_m \cap \text{cl } B_\varepsilon(x, E^n) = \emptyset$ and $|z_m - x| < \varepsilon$. Now $z_m = w + ta_{g_m}$ for some $w \in \text{co}(g_m)$ and $0 \leq t \leq \delta(g_m)$. If $w \in g_m$, then $|w - x| \leq (|w - x|^2 + t^2)^{1/2} = |z_m - x| < \varepsilon$ which contradicts the fact that $g_m \cap \text{cl } B_\varepsilon(x, E^n) = \emptyset$. Therefore $w \in \text{co}(g_m) \backslash g_m$ and by the definition of $\hat{g}_m$, $t = \delta(g_m)$. There are elements $w_1, w_2 \in g_m$ so that w lies in the line segment from w_1 to w_2. Let α be as shown in Fig. 1 and observe that $|w - x| \geq (\varepsilon^2 - \alpha^2)^{1/2}$ and $t = \delta(g_m) \geq |w_1 - w_2| \geq 2\alpha > \alpha$. Therefore, $|z_m - x| = (|w - x|^2 + t^2)^{1/2} > (\varepsilon^2 - \alpha^2 + \alpha^2)^{1/2} = \varepsilon$, which contradicts $|z_m - x| < \varepsilon$. Therefore, $\bar{g} \cap B_\varepsilon(x, E^{n+m+1}) = \emptyset$.

(C) Let $y \in E_g$ with $y \notin \bar{g}$. If $y \in \mathring{E}_g$ then it easily follows from the definition of $\bar{g}$ that there is an open set U in E_g with $y \in U$ and $U \cap \bar{g} = \emptyset$. If $y \in E^n$, then the same result easily follows from (B). Therefore, $\bar{g}$ is closed in E_g.

Proposition 2.2. $\bar{G}$ is an usc decomposition of E^{n+m+1}.

Proof. The previous proposition establishes that each $\bar{g} \in H_{\bar{G}}$ is compact. Let $\bar{N} = N_{\bar{G}} \cup E^n$ and let $\bar{H}$ be the restriction of $\bar{G}$ to $\bar{N}$; that is, $\bar{H}$ is the decomposition of $\bar{N}$ whose nondegenerate elements consist of precisely the nondegenerate elements of $\bar{G}$. We show that $\bar{N}$ is closed in E^{n+m+1} and $\bar{H}$ is an usc decomposition of $\bar{N}$, in which case $\bar{G}$ is the trivial extension to E^{n+m+1} of the usc decomposition $\bar{H}$ of the closed subset $\bar{N}$ so that $\bar{G}$ itself is usc.

Let $y_i \in \bar{g}_i$ and $y_i \to y \notin E^n$. Use Proposition 2.1(B) to show that there is a positive number M so that $g_i \cap B_M(0, E^n) \neq \emptyset$ for each i (choose $M > |y|$). Let $x_i \in g_i$ with

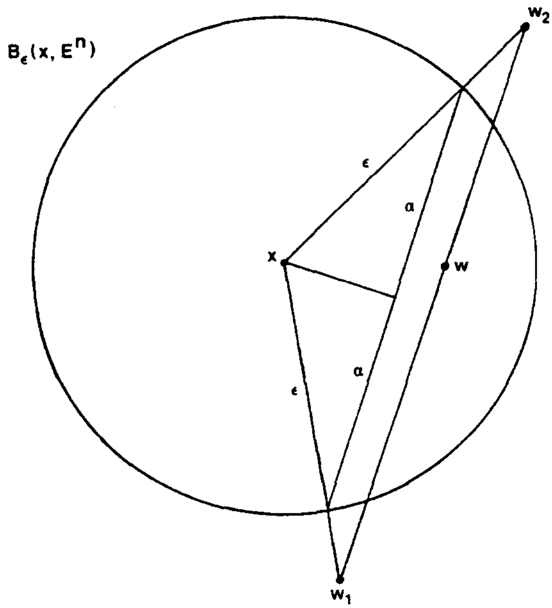

Fig. 1

$|x_i| < M$ and, by passing to a subsequence if necessary, assume that $x_i \to x$. Let $g = \pi^{-1}\pi(x)$ and observe that $\pi(g_i) = \pi(x_i) \to \pi(x) = \pi(g)$. Since $y \notin E^n$, it follows that $g \in H_G$ and $a_{g_i} \to a_g$, and since $y_i \to y$, it follows that $y \in \mathring{E}_g$. The definition of $\bar{g}_i$ and of $\bar{g}$ now shows that $y \in \bar{g} \subset \bar{N}$. Therefore $\bar{N}$ is closed.

To prove that $\bar{H}$ is an usc decomposition of $\bar{N}$, we need a preliminary Lemma:

For a given G-saturated open subset U of E^n, let $\bar{U} = \bigcup\{h \in \bar{H} \mid h \cap U \neq \emptyset\}$. Note that $\bar{U} \cap E^n = U$.

Lemma 2.3. *$\bar{U}$ is open in $\bar{N}$.*

Using Proposition 2.1(B) and some techniques in the proof of Proposition 2.1, the reader may verify that given $h \in \bar{H}$ and an open set V in E^{n+m+1} containing h, there is a G-saturated open neighborhood U of $h \cap E^n$ in E^n so that $h \subset \bar{U} \subset V$. Since $\bar{U}$ is an open $\bar{H}$-saturated subset of $\bar{N}$, $\bar{H}$ is usc.

Proof of Lemma 2.3. We show that $\bar{N} \backslash \bar{U}$ is closed. Let $\{y_i\}$ be a sequence in $\bar{N} \backslash \bar{U}$ with $y_i \to y$ and observe that since $\bar{N}$ is closed in E^{n+m+1}, $y \in \bar{N}$. There are two possibilities—either $y \in E^n$ or $y \notin E^n$. We leave it to the reader to use Proposition 2.1(B) to show that if $y \in E^n$, then $y \notin U = \bar{U} \cap E^n$. Assuming that $y \notin E^n$, we find elements $\bar{g}_i$, $\bar{g} \in H_{\bar{G}}$ with $y_i \in \bar{g}_i$ and $y \in \bar{g}$. By Proposition 2.1(A) we can write $y_i = x_i + t_i a_{g_i}$ and $y = x + t a_g$ for appropriate x_i, x, t_i, and $t > 0$ and as in the proof of Proposition 2.1(A), since $y_i \to y$ and $t \neq 0$, we have $\pi(g_i) \to \pi(g)$. Now it is clear that $y \notin \bar{U}$ for if $y \in \bar{U}$, then $g \subset U$ and since $\pi(g_i) \to \pi(g)$, eventually $g_i \subset U$ so that $y_i \in \bar{U}$, a contradiction.

We have produced an usc decomposition $\bar{G}$ of E^{n+m+1} which intersects down to E^n to yield our original decomposition G. There is little hope in showing that $\bar{G}$ is shrinkable since, in general, $\bar{G}$ is not cell-like. However, without giving up upper semicontinuity, we modify $\bar{G}$ so as to obtain a cell-like decomposition $\tilde{G}$ of E^{n+m+1}.

Let $g \in H_G$ and let $y = x + ta_g \in \bar{g}$ where $x \in \mathrm{co}(g)$ and $0 \leq t \leq \delta(g)$ (Proposition 2.1(A)). Let $A[y]$ denote the straight line segment in E_g from y to the unique element in $\mathrm{co}(g) + \delta(g)a_g$ 'above' y; that is, $A[y] = \{x + sa_g \mid t \leq s \leq \delta(g)\}$. Let $\tilde{g} = \bigcup \{A[y] \mid y \in \bar{g}\} \subset \mathrm{co}(g) + A_g$ and note that $\bar{g} \subset \tilde{g} \subset E_g$ and $\tilde{g} \cap E^n = g$ (see Fig. 2). It is easy to see that $\tilde{g}$ is compact (closed and bounded) and, since $\tilde{g}$ obviously is contractible in itself (first contract in the a_g direction to the $\delta(g)$ level, then contract $\mathrm{co}(g)$), $\tilde{g}$ is cell-like. The analogs of Proposition 2.1(B) and Lemma 2.3 hold for $\tilde{g}$ in place of $\bar{g}$ and these will be used in showing that the cell-like decomposition $\tilde{G}$ of E^{n+m+1} defined by $H_{\tilde{G}} = \{\tilde{g} \mid g \in H_G\}$ is usc.

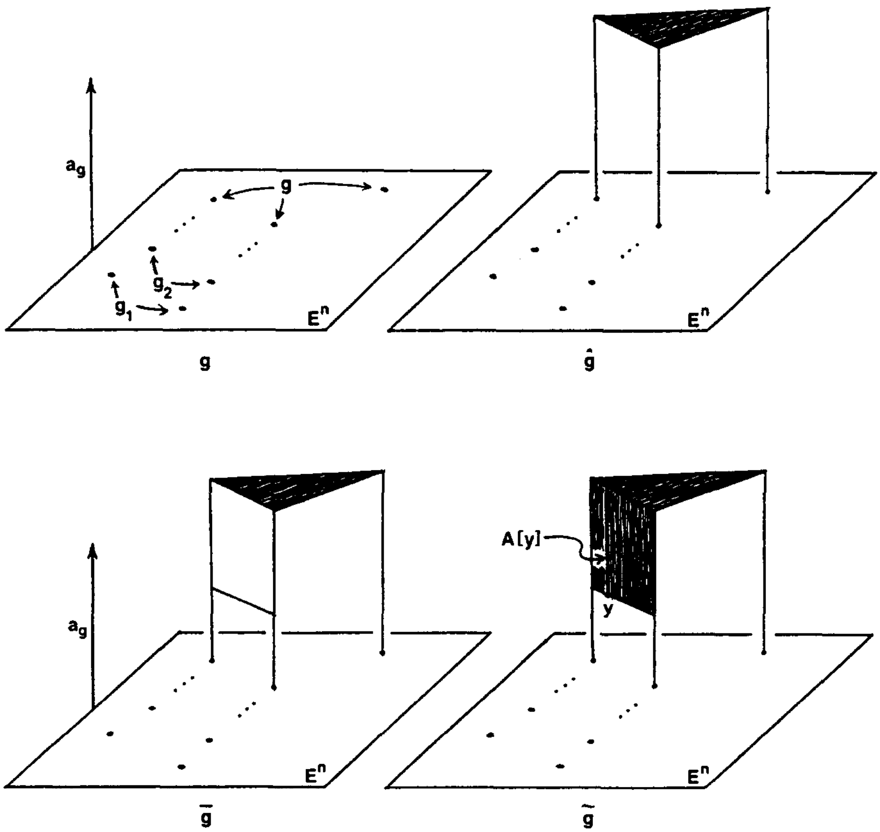

Fig. 2

Lemma 2.4. (A) *Proposition* 2.1(B) *holds with* $\tilde{g}$ *in place of* $\bar{g}$. (B) *Let* U *be a* G*-saturated open subset of* E^n *and define* $\tilde{U} = \bigcup \{h \in \tilde{G} \mid h \cap U \neq \emptyset\}$. *Then* $\tilde{U}$ *is open in* $\tilde{N} = N_{\tilde{G}} \cup E^n$.

Proof. (A) This is a Corollary of the construction of $\tilde{g}$ from $\bar{g}$ and of Proposition 2.1(B).

(B) We show that $\tilde{N} = N_{\tilde{G}} \cup E^n$ is a closed subset of E^{n+m+1} in Proposition 2.5. Using this fact, the proof of (B) is exactly like that of Lemma 2.3 except that Lemma 2.4(A) is used in place of Proposition 2.1(B).

We are ready to state the Main Result of this Section.

Proposition 2.5. $\tilde{G}$ *is a finite dimensional cell-like usc decomposition of* E^{n+m+1}.

Proof. We have already observed that $\tilde{G}$ is cell-like. In proving that $\tilde{G}$ is usc, we follow the proof of Proposition 2.2. Let $\tilde{H}$ denote the decomposition of $\tilde{N} = N_{\tilde{G}} \cup E^n$ with $H_{\tilde{H}} = H_{\tilde{G}}$. We show that $\tilde{N}$ is closed in E^{n+m+1} and $\tilde{H}$ is an usc decomposition of $\tilde{N}$. Once we know that $\tilde{G}$ is usc, it follows that $\tilde{\pi}(N_{\tilde{G}})$ is naturally homeomorphic to $\pi(N_G)$ so that $\tilde{G}$ is finite dimensional. (Here, $\tilde{\pi}$ is the decomposition map for $\tilde{G}$.)

Let $w_i \in \tilde{g}_i$ and $w_i \to w \notin E^n$. Exactly as in the proof that $\bar{N}$ is closed (Proposition 2.2), using Lemma 2.4(A) in place of Proposition 2.1(B), there is some $g \in H_G$ so that $\pi(g_i) \to \pi(g)$. Let $y_i \in \bar{g}_i$ with $w_i \in A[y_i]$. Then without loss of generality, $y_i \to y$ for some $y \in \bar{N}$. Let $\bar{\pi}$ be the decomposition map for $\bar{G}$ and observe that since $\pi(g_i) \to \pi(g)$, $\bar{\pi}(\bar{g}_i) \to \bar{\pi}(\bar{g})$. Since $y_i \in \bar{g}_i$ and $y_i \to y$, we have that $\bar{\pi}(\bar{g}_i) \to \bar{\pi}(y)$ so that $y \in \bar{g}$. It is now easy to show that $w \in A[y] \subset \tilde{g} \subset \tilde{N}$ and thus $\tilde{N}$ is closed.

We leave it to the reader to use Lemma 2.4(A) to show that if $h \in \tilde{H}$ and V is an open set in E^{n+m+1} containing h, then there is a G-saturated open neighborhood U of $h \cap E^n$ in E^n so that $h \subset \tilde{U} \subset V$. Since $\tilde{U}$ is an open $\tilde{H}$-saturated subset of $\tilde{N}$, $\tilde{H}$ is usc.

3. Main results

$\tilde{G}$ denotes the cell-like usc decomposition of E^{n+m+1} constructed above from a given usc decomposition G of E^n and a given embedding ψ of $\pi(N_G)$ into the m-sphere S^m.

Theorem 3.1. $\tilde{G}$ *is shrinkable provided one of the following conditions is satisfied*: (i) $m \geqslant 3$ *or* (ii) $m = 2$, $n \geqslant 2$, *and* G *is 1-dimensional.*

As an immediate corollary we have

Corollary 3.2. *If G is an* usc *decomposition of E^n and $\pi(N_G)$ embeds in S^m where $m \geq 3$, then E^n/G embeds as a closed subset of E^{n+m+1}. If G is a 1-dimensional* usc *decomposition of $E^n (n \geq 2)$ and $\pi(N_G)$ embeds in S^2, then E^n/G embeds as a closed subset of E^{n+3}.*

Corollary 3.2 provides at least a partial answer to the question posed in the introduction.

Corollary 3.3. *Let G be a k-dimensional $(k \geq 0)$* usc *decomposition of $E^n (n \geq 1)$. Then E^n/G embeds as a closed subset of E^{n+2k+2}.*

Proof. If $k \geq 1$, then $2k+1 \geq 3$ and since $\pi(N_G)$ embeds in S^{2k+1}, Corollary 3.2 implies that E^n/G embeds as a closed subset of E^{n+2k+2}. The proof of the 0-dimensional case requires a separate argument which is included in an appendix at the end of the paper.

We discuss the sharpness of Corollary 3.3 in Section 5.

Corollary 3.4. *Let X be a space that embeds as a closed subset of E^n and let G be an* usc *decomposition of X such that $\pi(N_G)$ embeds in $S^m (m \geq 3)$. Then X/G embeds as a closed subset of E^{n+m+1}. In particular, X/G embeds in E^{n+2k+2} if G is k-dimensional.*

4. Proof of the Main Theorem

We apply the Cell-like Approximation Theorem in order to prove Theorem 3.1. Since $n+m+1 \geq 5$ and $E^{n+m+1}/\tilde{G}$ is obviously finite dimensional, it suffices to show that $E^{n+m+1}/\tilde{G}$ satisfies the DDP. The strategy for doing this is as follows. First, we decompose $\psi(\pi(N_G))$ into countably many compact subsets $C_1, C_2, \ldots$ and a subset L so that $C_i \cap L = \emptyset$ for each i and the embedding dimension of each C_i and of each compact subset of L is $\leq m-2$ in S^m. Then given maps f_1 and f_2 of B^2 into $E^{n+m+1}/\tilde{G}$, we produce approximate lifts $F_1, F_2 : B^2 \to E^{n+m+1}$. By successive moves, we move F_1 off of each $N_{\tilde{G}_i} = \{\tilde{g} \mid \psi(\pi(g)) \in C_i\}$ so that in the limit we obtain a map $\mu_1' : B^2 \to E^{n+m+1}/\tilde{G}$ approximating f_1 and with the property that $\mu_1'(B^2)$ contains no point $\tilde{\pi}(\tilde{g})$ if $\psi(\pi(g)) \in \bigcup_{i=1}^{\infty} C_i$ where $\tilde{\pi}: E^{n+m+1} \to E^{n+m+1}/\tilde{G}$ is the decomposition map. We then adjust F_2 to obtain F_2' so that $F_2'(B^2)$ touches no $\tilde{g} \in H_{\tilde{G}}$ if $\tilde{\pi}(\tilde{g}) \in \mu_1'(B^2)$. Then μ_1' and $\mu_2' = \tilde{\pi} F_2'$ satisfy $\mu_1'(B^2) \cap \mu_2'(B^2) \cap \tilde{\pi}(N_{\tilde{G}}) = \emptyset$ and a final general position adjustment in the manifold part of the decomposition space produces the desired disjoint approximations.

We use some results from the theory of embedding dimension of compacta in S^m. The reader is referred to [1, Chapter 21] and [3] for standard results about embedding dimension.

Definition 4.1. A compact subset X of S^m has *embedding dimension* $\leqslant k$, written $\operatorname{dem} X \leqslant k$, if each $(m-k-1)$-dimensional (tamely embedded) polyhedron P in S^m can be moved off of X by an arbitrarily small ambient isotopy of S^m, with support arbitrarily close to $X \cap P$. As usual, $\operatorname{dem} X = k$ provided $\operatorname{dem} X \leqslant k$ but not $\operatorname{dem} X \leqslant k-1$.

Compact subsets of S^m of embedding dimension k in many ways behave like compact k-dimensional subpolyhedra of S^m. This point of view is reinforced by the following general position Theorem which is used later.

Thoerem 4.2. [1, Proposition 21.11]. *Let X and Y be compact subsets of S^m and let $\varepsilon > 0$. Then there is an ε-homeomorphism h of S^m so that*

$$\operatorname{dem}[h(X) \cap Y] \leqslant \operatorname{dem} X + \operatorname{dem} Y - m.$$

In particular, since $\operatorname{dem} P = \dim P$ whenever P is a subpolyhedron of S^m and since embedding dimension is greater than or equal to dimension, we have the conclusion

$$\dim[h(X) \cap P] \leqslant \operatorname{dem} X + \dim P - m$$

in case P is a subpolyhedron of S^m. A standard proof of this fact involves successively moving X off of the $(m - \operatorname{dem} X - 1)$-skeleta of finer and finer triangulations of P by space homeomorphisms of S^m so that in the limit one obtains a homeomorphism h of S^m which pushes X off of each $(m - \operatorname{dem} X - 1)$-skeleton. Then $h(X) \cap P$ retracts to the dual $[\dim P - (m - \operatorname{dem} X - 1) - 1]$-skeleton of each triangulation so that $\dim[h(X) \cap P] \leqslant \dim P - (m - \operatorname{dem} X - 1) - 1 = \operatorname{dem} X + \dim P - m$.

Let C be any compact subset of $\psi(\pi(N_G))$ of embedding dimension $\leqslant m-2$ in S^m and let $\tilde{\mathscr{G}}$ denote the usc decomposition of E^{n+m+1} induced from $\tilde{G}$ by C. That is, $\tilde{\mathscr{G}}$ is the decmposition of E^{n+m+1} whose nondegenerate elements are precisely those $\tilde{g} \in H_{\tilde{G}}$ such that $\psi(\pi(g)) \in C$. In this Section, ρ denotes the metric for E^{n+m+1} and $\tilde{\rho}$ for $E^{n+m+1}/\tilde{G}$ while $\tilde{\pi}$ is the decomposition map for $\tilde{G}$. The following Lemma holds in case $n \geqslant 1$ and $m \geqslant 3$ or in case $n \geqslant 2$ and $m = 2$.

Lemma 4.2. *Let $\varepsilon, \alpha > 0$. Then any map $F: B^2 \to E^{n+m+1}$ can be approximated by a map $\mu: B^2 \to E^{n+m+1}$ so that $\tilde{\rho}(\tilde{\pi}F, \tilde{\pi}\mu) < \varepsilon$, $\mu(B^2) \cap N_{\tilde{\mathscr{G}}} = \emptyset$, and $\rho(\mu(B^2), E^n) \geqslant \rho(F(B^2), E^n) - \alpha$.*

The author is indebted to R.J. Daverman for suggesting the use of embedding dimension as a condition to restrict C in lieu of the author's original condition in Lemma 4.2. This allowed for a more unified and cohesive treatment of the Main Results.

Proof of Lemma. Without loss of generality, we may assume that F is a *p.l.*

embedding in general position with respect to E^n so that $F(B^2)\cap E^n=\emptyset$ (since $n+m+1\geq 5$). Then $F(B^2)\subset E^n\times S^m\times(0,\infty)$ and we write $F=(F_1,F_2,F_3)$ where F_i is F followed by projection to the ith factor of $E^n\times S^m\times(0,\infty)$ for $i=1,2,3$. Approximate F_2 by a *p.l.* map f_2 so that f_2 embeds each simplex of some sufficiently fine finite triangulation T of B^2. Then $f_2(B^2)$ is a 2-dimensional polyhedron in S^m and we use the general position theorem for embedding dimension (Theorem 4.2) to produce an approximation $\mu_2=hf_2$ of f_2 with $\dim[\mu_2(B^2)\cap C]\leq 0$ where h is a homeomorphism of S^m. Since μ_2 embeds each simplex of T, $\dim\mu_2^{-1}(C)\leq 0$.

Let $u=(F_1,\mu_2,F_3)$ and let $A=u^{-1}(N_{\tilde{\mathcal{G}}})$ so that A consists of all points $b\in B^2$ such that $u(b)\in\tilde{g}$ for some $\tilde{g}\in H_{\tilde{\mathcal{G}}}$. Then $A\subset\mu_2^{-1}(C)$ and therefore A is a compact 0-dimensional subset of B^2. Let $b\in A$ and suppose that $u(b)\in A[y]\subset\tilde{g}\in H_{\tilde{\mathcal{G}}}$ where $y\in\tilde{g}$ (see the definition of $\tilde{g}$). Given an arbitrary $\tilde{G}$-saturated open neighborhood U of $\tilde{g}$, there are open sets $V\subset E^n$ and $W\subset S^m$ and a closed interval $[\beta,\gamma]$ with $\delta(g)<\gamma$ so that $A[y]\subset V\times W\times[\beta,\gamma]\subset U$ and so that $\delta(g')<\gamma$ for any $g'\in H_G$ with $\psi(\pi(g'))\in W\cap C$. The only difficulty here is in choosing W. First choose V, W', and $[\beta,\gamma]$ so that $\delta(g)<\gamma$ and $A[y]\subset V\times W'\times[\beta,\gamma]\subset U$ and note that $\psi(\pi(g))\in W'\cap C$. Since G is usc and $\delta(g)<\gamma$, there is a small neighborhood $W\subset W'$ such that $\delta(g')<\gamma$ for every $g'\in H_G$ with $\psi(\pi(g'))\in W\cap C$. Now let D be any 2-cell in B^2 with $\partial D\cap\mu_2^{-1}(C)=\emptyset$ and $u(D)\subset V\times W\times[\beta,\gamma]$. Since $\mu_2^{-1}(C)$ is compact, there is another 2-cell $D'\subset\operatorname{int}D$ with $\mu_2^{-1}(C)\cap D\subset\operatorname{int}D'$. Let $\nu: D\to[0,1]$ be a Urysohn function with $\nu(\partial D)=0$ and $\nu(D')=1$ and define $v_D: D\to E^{n+m+1}$ via $v_D(x)=(F_1(x),\mu_2(x),\nu(x)\gamma+(1-\nu(x))F_3(x))$ for $x\in D$. Note that $v_D|\partial D=u|\partial D$. It is clear that $v_D(D)\subset V\times W\times[\beta,\gamma]\subset U$ and it is easy to show that $v_D(D)\cap N_{\tilde{\mathcal{G}}}=\emptyset$. (Recall that Proposition 2.1(A) and the definition of $\tilde{h}$ imply that $\tilde{h}\subset\operatorname{co}(h)+A_h=\operatorname{co}(h)\times\{\psi(\pi(h))\}\times[0,\delta(h)]$ for $\tilde{h}\in\tilde{N}_{\tilde{G}}$.)

We have shown that for each $b\in A$ and each $\delta>0$, there is an $\eta>0$ so that if D is a 2-cell contained in the η-neighborhood of b and whose boundary misses $\mu_2^{-1}(C)$, then there is a map $v_D: D\to E^{n+m+1}$ so that $\tilde{\rho}(\tilde{\pi}u(x),\tilde{\pi}v_D(x))<\delta$ for all $x\in D$, $v_D|\partial D=u|\partial D$, and $v_D(D)\cap N_{\tilde{\mathcal{G}}}=\emptyset$. Furthermore, $\rho(v_D(x),E^n)\geq\rho(u(x),E^n)$.

Assume that all the previous general position moves and approximations have been so small that $\tilde{\rho}(\tilde{\pi}F,\tilde{\pi}u)<\varepsilon/2$ and $\rho(F,u)<\alpha/2$. Since A is compact 0-dimensional we can find finitely many pairwise disjoint 2-cells $D_1,\ldots,D_j$ whose boundaries miss $\mu_2^{-1}(C)$ and which cover A and maps v_{D_i} for $i=1,\ldots,j$ as above. The maps v_{D_i} are forced to satisfy $\tilde{\rho}(\tilde{\pi}u(x),\tilde{\pi}v_{D_i}(x))<\varepsilon/2$. Let $\mu: B^2\to E^{n+m+1}$ be defined by $\mu|D_i=v_{D_i}$ for $i=1,.,\ldots,j$ and $\mu=u$ otherwise. Then μ satisfies the conclusions of the lemma.

Remark. Lemma 4.2 is much easier to prove in case dem $C\leq m-3$ since we can let $\mu=(F_1,\mu_2,F_3)$ where μ_2 is a general position move of F_2 so that $\mu_2(B^2)\cap C=\emptyset$. This is all we need to prove Theorem 3.1 in case $m\geq 5$. It is the inclusion of the low dimensional cases in Theorem 3.1 which requires dem $C\leq m-2$ and therefore requires a more difficult proof for the lemma.

We now show that we can decompose $\psi(\pi(N_G))$ into countably many compact subsets $C_1, C_2, \ldots$ and a subset L so that $C_i \cap L = \emptyset$ for each i and the embedding dimension of each C_i and of each compact subset of L is $\leqslant m-2$. If $m \geqslant 3$, $m = 2r$ or $m = 2r+1$ for some $r \geqslant 1$. Write $\psi(\pi(N_G)) = \bigcup_{i=1}^{\infty} \sigma_i$ where each σ_i is compact ($\pi(N_G)$ is a σ-compactum since $N_G = \bigcup_{i=1}^{\infty} N_G(1/i)$ is an F_σ subset of the σ-compact space E^n) and let T_j be a sequence of triangulations of S^m with mesh $T_j \to 0$. Let $C_{ij} = \sigma_i \cap T_j^{(r)}$ and $L = \psi(\pi(N_G)) \setminus \bigcup \{C_{ij} \mid \text{all } i \text{ and } j\}$ and rename the C_{ij} to obtain the C_i. Then each C_i is compact and $\dim C \leqslant r \leqslant m-2$ so that $\operatorname{dem} C \leqslant m-2$ where C is any C_i or any compact subset of L. Note that $r \leqslant m-3$ if $m \geqslant 5$. If $m = 2$ and $\dim \pi(N_G) = 1$, a simpler argument yields the desired decomposition.

For each i, let $\tilde{G}_i$ be the usc decomposition of E^{n+m+1} induced from C_i by $\tilde{G}$. We are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $f_1, f_2 : B^2 \to E^{n+m+1}/\tilde{G}$ be maps and let $\varepsilon > 0$. Choose approximate lifts $F_1, F_2 : B^2 \to E^{n+m+1}$ for f_1, f_2 so that $\tilde{\rho}(\tilde{\pi} F_i, f_i) \leqslant \varepsilon/3$ for $i = 1, 2$ and assume that $F_1(B^2) \cap E^n = \emptyset$ (by general position, since $n+2 < n+m+1$). Let $\eta = (1/2) \cdot \rho(F_1(B^2), E^n)$ and let $U = B_\eta(E^n, E^{n+m+1}) \setminus N_{\tilde{G}(\eta)}$ so that U is a $\tilde{G}$-saturated open subset of E^{n+m+1} containing all $\tilde{g} \in N_{\tilde{G}}$ of diameter less than η. By Lemma 4.2, we know that for any $\delta > 0$ and for each i, we can find a map $g : B^2 \to E^{n+m+1}$ satisfying (i) $\tilde{\pi} g(B^2) \cap \tilde{\pi}(N_{\tilde{G}_i}) = \emptyset$, (ii) $\tilde{\rho}(\tilde{\pi} g, \tilde{\pi} F_1) < \delta$, and (iii) $\rho(g(B^2), E^n) > \eta$. Using (i) and (ii) in conjunction with a well-known limit argument, since each $\tilde{\pi}(N_{\tilde{G}_i})$ is compact, we can successively move $\tilde{\pi} F_1$ off of each $\tilde{\pi}(N_{\tilde{G}_i})$ by moves so small that we obtain a limit map μ_1' with $\tilde{\rho}(\mu_1', \tilde{\pi} F_1) < \varepsilon/3$ and $\mu_1'(B^2) \cap \tilde{\pi}(\bigcup_{i=1}^{\infty} N_{\tilde{G}_i}) = \emptyset$. Since $\tilde{\pi}(U)$ is open, (iii) implies that μ_1' can be chosen so that $\mu_1'(B^2) \cap \tilde{\pi}(U) = \emptyset$. As a consequence, if $\tilde{g} \in H_{\tilde{G}}$ and $\tilde{\pi}(\tilde{g}) \in \mu_1'(B^2)$ then $\psi(\pi(g)) \notin \bigcup_{i=1}^{\infty} C_i$ and $\delta(\tilde{g}) \geqslant \eta$. Therefore, the set $C = \{\psi(\pi(g)) \in \psi(\pi(N_G)) \mid \tilde{\pi}(\tilde{g}) \in \mu_1'(B^2)\} \approx \mu_1'(B^2) \cap \tilde{\pi}(N_{\tilde{G}(\eta)})$ is a compact subset of L and the embedding dimension of C is $\leqslant m-2$ in S^m. Let $\tilde{\mathscr{G}}$ be the decomposition of E^{n+m+1} induced from $\tilde{G}$ by C. Lemma 4.2 applies to give a map $\mu_2' : B^2 \to E^{n+m+1}/\tilde{G}$ so that $\mu_2'(B^2) \cap \tilde{\pi}(N_{\tilde{\mathscr{G}}}) = \emptyset$ and $\tilde{\rho}(\mu_2', \tilde{\pi} F_2) < \varepsilon/3$. Observe that

$$\mu_1'(B^2) \cap \mu_2'(B^2) \cap \tilde{\pi}(N_{\tilde{G}}) = \emptyset$$

and since $\mu_1'(B^2) \cap \tilde{\pi}(U) = \emptyset$, $\mu_1'(B^2) \cap \mu_2'(B^2) \subset (E^{n+m+1}/\tilde{G}) \setminus \tilde{\pi}(E^n)$, an $(n+m+1)$-manifold. A final general position adjustment in the manifold part of the decomposition space yields the desired disjoint approximations μ_1 and μ_2.

5. Examples

For $2k+1 > n > 1$, dimension theory provides a sharpening of Corollary 3.3 if G is *closed* k-dimensional. In this case, $\dim E^n/G = m = \max\{n, k\}$ and E^n/G embeds in E^{2m+1} ($2m+1 < n+2k+2$ since $2k+1 > n$). As mentioned in the introduction, E^n/G embeds in E^{n+1} provided G is cell-like and $n \geqslant 4$. An affirmative answer to the following question would completely answer the question posed in the introduction.

Does there exist a (nonclosed and non-cell-like) k-dimensional usc decomposition G of E^n for which E^n/G does *not* embed in E^{n+2k+1}?

In [5], Gillman shows that for each integer m, $m \geq 1$, there exists a decomposition of $S^{2m-1}$with nondegenerate elements consisting of five $(m-1)$-spheres such that the associated decomposition space cannot be embedded in S^{2m}. This provides an affirmative answer to the above question in case $k=0$ and n is odd. More recently, R.J. Daverman has provided an affirmative answer to the question in case k is an arbitrary non-negative integer and n is odd. His examples, which we construct below, not only show that Corollary 3.3 is sharp in case n is odd, but also that Corollary 3.3 is off by no more than one dimension in case n is even.

Example 5.1. Let $k \geq 0$ be an integer and let $n=2m+1$ or $2m+2$. Then there is a k-dimensional usc decomposition G of E^n whose decomposition space E^n/G does not embed in $E^{2m+2k+2}$.

The example is constructed so that E^n/G contains a copy of Flores' example [4] of an $(m+k+1)$-dimensional complex that does not embed in $E^{2m+2k+2}$.

Let L_0 be Flores' $(m+k+1)$-dimensional complex and for $r \geq 1$, recursively define L_r to be the star of the m-skeleton of L_{r-1} in the second barycentric subdivision of L_{r-1} and define D_r to be the dual k-skeleton of L_{r-1}. Thus $D_{r+1} \subset L_r = st(L_{r-1}^{(m)}, \beta^2 L_{r-1}) \subset \beta^{2r} L_0$. We use geometric barycenters so that mesh $L_r \to 0$ as $r \to \infty$. There is a standard surjective *p.l.* map $h_r: L_r \to L_{r-1}$ which is one to one over precisely $L_{r-1} \backslash D_r$ and which is limited by βL_{r-1} (that is, h_r maps each simplex σ of L_r into a simplex ω of βL_{r-1}). The h_r's can be chosen so carefully that for each $s<r$, $h_{s+1} \circ \cdots \circ h_r$ is limited by $\beta^{r-s} L_s$. Let L_∞ denote the limit of the inverse sequence $\{L_r, h_r\}$ of compact polyhedra. The following diagram induces a map $h: L_\infty \to L_0$.

$$\begin{array}{ccccccccc}
L_0 & \xleftarrow{h_1} & L_1 & \xleftarrow{h_2} & L_2 & \xleftarrow{h_3} & L_3 & \xleftarrow{h_4} & \cdots \\
{\scriptstyle id}\downarrow & & {\scriptstyle h_1}\downarrow & & {\scriptstyle h_1 h_2}\downarrow & & {\scriptstyle h_1 h_2 h_3}\downarrow & & \\
L_0 & = & L_0 & = & L_0 & = & L_0 & = & \cdots
\end{array}$$

It is obvious that h is surjective since h is given by $h(x_0, x_1, \ldots) = (x_0, x_0, \ldots) = x_0$ and it is straightforward to show that h is one to one over $L_0 \backslash \mathscr{D}$ where $\mathscr{D} = D_1 \cup h_1(D_2) \cup h_1 h_2(D_3) \cup \cdots$. Furthermore, L_∞ is a compact m-dimensional space. This follows from classical characterizations of the dimensions of compacta in terms of properties of inverse sequences determining the compacta (see for example [7, Thoerem 4.1]).

Let G_∞ denote the usc decomposition of L_∞ induced by h. Since h is surjective we can write $L_0 = h(N_{G_\infty}) \cup h(L_\infty \backslash N_{G_\infty})$. Since L_0 is $(m+k+1)$-dimensional and

$h(L_\infty\backslash N_{G_\infty})$ is less than or equal to m-dimensional, classical dimension theory [6, p. 28] implies that G_∞ is greater than or equal to k-dimensional. Since h is one to one over $L_0\backslash\mathcal{D}$ and $\mathcal{D}$ is k-dimensional, G_∞ must be less than or equal to k-dimensional (by definition). Thus G_∞ is a k-dimensional usc decomposition of the m-dimensional space L_∞. (Actually, h is one to one over precisely $L_0\backslash\mathcal{D}$ so that immediately G_∞ is seen to be k-dimensional.)

Since L_∞ is m-dimensional, L_∞ embeds in E^n ($n=2m+1$ or $2m+2$) and we let G denote the trivial extension to E^n of the image of G_∞ under such an embedding. Since L_∞ is compact, G is an usc decomposition of E^n. G is k-dimensional since G_∞ is and E^n/G contains a copy of L_0 since $L_\infty/G_\infty\approx L_0$.

Question 5.2. If n is even, does there exist a k-dimensional usc decomposition G of E^n for which E^n/G does not embed in E^{n+2k+1}?

Appendix

We prove that if G is a 0-dimensional usc decomposition of E^n, then E^n/G embeds as a closed subset of E^{n+2}. The argument employs an amalgamation technique in conjunction with 'by hands' partial shrinking of amalgamated decompositions. The author is indebted to J.J. Walsh for suggesting this approach to the problem.

We begin with a special case by supposing that cl N_G is compact so that we can cover $\pi(N_G)$ by open sets $V\in\mathcal{V}_1$ with the following properties: $\mathcal{V}_1$ is a countable cover of $\pi(N_G)$ by sets open in E^n/G so that each cl V is compact and so that $\bar{\mathcal{V}}_1=\{\text{cl } V: V\in\mathcal{V}_1\}$ is pairwise disjoint. In particular, $Fr\, V\cap\pi(N_G)=\emptyset$ for each $V\in\mathcal{V}_1$ and we assume further that $V\cap\pi(N_G)\neq\emptyset$ for each such V. Let $\mathcal{U}_i=\{V\in\mathcal{V}_1 \mid \exists g\in G \text{ with } \delta(g)\geq 1/2^i \text{ and } \pi(g)\in V\}$ $(i=1,2,\dots)$ and define $\mathcal{V}_{(1,1)}=\mathcal{U}_1$ and $\mathcal{V}_{(1,i)}=\mathcal{U}_i\backslash\mathcal{U}_{i-1}$ $(i=2,3,\dots)$. Then we assume that $\mathcal{V}_{(1,i)}$ is finite for each i and for $i=2,3,\dots$, if $V\in\mathcal{V}_{(1,i)}$, then diam cl $V<1/2^i$ and $\delta(\pi^{-1}(\text{cl } V))<1/2^{i-2}$.

Let G_1 be the decomposition of E^n with $H_{G_1}=\{\pi^{-1}(\text{cl } V)\mid V\in\mathcal{V}_1\}$. Then G_1 is a null sequence decomposition of E^n and G_1 is an amalgamation of G (for each $g\in G$ there is a $g_1\in G_1$ so that $g\subset g_1$). We write $G<G_1$ to denote this. We now construct a null sequence decomposition $\tilde{G}_1$ of E^{n+2} into starlike sets. For each $g\in G_1$, let $x_g\in\text{co}(g)$ and choose $b_g\in S^1$ and an interval (θ_g,ψ_g) (angular measure) about b_g in S^1 so that $\{(\theta_g,\psi_g)\mid g\in H_{G_1}\}$ is pairwise disjoint. Let $c_g=x_g+\delta(g)b_g\in E^n\times E^2$ and let $\tilde{g}$ be the geometric cone of g in E^{n+2} with cone point c_g. Note that $\delta(\tilde{g})<3\cdot\delta(g)$. Let $\tilde{G}_1$ be the decomposition of E^{n+2} with $H_{\tilde{G}_1}=\{\tilde{g}\mid g\in H_{G_1}\}$.

$\tilde{G}_1$ is an usc decomposition of E^{n+2} into points and countably many starlike sets and therefore is strongly shrinkable, that is, for each open set W containing $N_{\tilde{G}_1}$ there is a shrinking homeomorphism fulfilling the shrinkability criterion that keeps each point of $E^{n+2}\backslash W$ fixed. Let h_1 be a homeomorphism of E^{n+2} shrinking $\tilde{G}_1$ to $\frac{1}{4}$-size ($\delta(h_1(\tilde{g}))<\frac{1}{4}$ for each $\tilde{g}$).

Choose an open cover $\mathcal{V}_2$ of $\pi(N_G)$ similar to $\mathcal{V}_1$ with $\bar{\mathcal{V}}_2 < \mathcal{V}_1$. Define $\mathcal{V}_{(2,i)}$ similarly to $\mathcal{V}_{(1,i)}$ and assume that $\mathcal{V}_2$ is chosen so that each $\mathcal{V}_{(2,i)}$ is finite and for $V \in \mathcal{V}_{(2,i)}$, diam cl $V < 1/2^{i+1}$ for $i = 1, 2, \ldots$ and $\delta(\pi^{-1}(\text{cl } V)) < 1/2^{i-2}$ for $i = 2, 3, \ldots$. Let G_2 be the decomposition of E^n with $H_{G_2} = \{\pi^{-1}(\text{cl } V) \mid V \in \mathcal{V}_2\}$. G_2 is a null sequence decomposition of E^n and $G < G_2 < G_1$.

We now construct, with certain controls, a null sequence decomposition $\tilde{G}_2$ of E^{n+2} into starlike sets. For each $g \in H_{G_1}$, let U_g be an open neighborhood of $\tilde{g}$ so that $\delta(h_1(U_g)) < \frac{1}{4}$. Choose $\theta_g < \theta'_g < \psi'_g < \psi_g$ so that for each $y \in (\theta'_g, \psi'_g)$, the cone of g in E^{n+2} with cone point $x_g + \delta(g)y$ lies in U_g. Let $J_g = \{j \in H_{G_2} \mid j \subset g\}$ and let $\eta = \rho(g, E^{n+2} \setminus U_g)$. For each $j \in J_g$, choose $b_j \in (\theta'_g, \psi'_g)$ and 'buffer' sets (θ_j, ψ_j) as before so that $\{(\theta_j, \psi_j) \mid j \in J_g\}$ is pairwise disjoint and so that $\theta'_g < \theta_j < \psi_j < \psi'_g$. Let $x_j \in \text{co}(j)$. If $\delta(j) \geq \eta/3$, let $c_j = x_g + \delta(g) b_j$ and if $\delta(j) < \eta/3$, let $c_j = x_j + \delta(j) b_j$. Let $\tilde{j}$ be the cone of j in E^{n+2} with cone point c_j and observe that $\tilde{j} \subset U_g$ for each $j \in J_g$. Let $\tilde{G}_2$ be the decomposition of E^{n+2} with $H_{\tilde{G}_2} = \{\tilde{j} \mid j \in H_{G_2}\}$. Then $\tilde{G}_2$ is a null sequence decomposition of E^{n+2} into starlike sets and $h_1(\tilde{G}_2)$ is a null sequence decomposition of E^{n+2} into starlike equivalent sets and therefore is strongly shrinkable.

For each $g \in H_{G_1}$ and $j \in J_g$, choose an open neighborhood W_j of $\tilde{j}$ with the following properties: (i) $W_j \subset U_g$, (ii) given $g' \neq g$ and $j' \in J_{g'}$, $W_j \cap W_{j'} = \emptyset$, (iii) W_j is $\tilde{G}_2$-saturated, and (iv) $W_j \cap E^n \subset \text{int } g$. Let $\mathcal{W}_2 = \{W_j \mid j \in H_{G_2}\}$ and let $W^2 = \bigcup \{W_j \mid j \in H_{G_2}\}$. Let h_2 be a homeomorphism of E^{n+2} satisfying: (a) $h_2 \mid E^{n+2} \setminus h_1(W^2)$ is the identity, (b) h_2 is limited by $h_1(\mathcal{W}_2)$, (c) h_2 shrinks $h_1(\tilde{G}_2)$ to $\frac{1}{8}$-size, and (d) $\rho(h_2, \text{id}) < \frac{1}{2}$.

We continue in this manner making sure that when we choose U_j for $j \in H_{G_2}$, we have $U_j \subset W_j$. After countably many steps, we obtain the following:

1) A sequence $G_1, G_2, \ldots$ of null sequence decompositions of E^n with $G < \cdots < G_k < \cdots < G_2 < G_1$ so that if $g \in G$ and $g \subset g_k \in Gk$ $(k = 1, 2, \ldots)$ then $g = \bigcap_{k=1}^{\infty} g_k \in G$.

2) A sequence $\tilde{G}_1, \tilde{G}_2, \ldots$ of null sequence decompositions of E^{n+2} into starlike sets with $\tilde{G}_k \cap E^n = G_k$.

3) For each $k = 2, 3, \ldots$ a G_k-saturated cover $\mathcal{W}_k = \{W_j \mid j \in H_{G_k}\}$ of $N_{\tilde{G}_k}$ satisfying $\mathcal{W}_{k+1} < \mathcal{W}_k$ and (i) $\tilde{j} \subset W_j$, (ii) $W_j \cap W_{j'} = \emptyset$ if $j \subset g$, $j' \subset g'$ and $g \neq g'$ for $g, g' \in H_{G_{k-1}}$ and (iii) if $j \in H_{G_{k+1}}$, $g \in H_{G_k}$, and $j \subset g$, then $W_j \cap E^n \subset \text{int } g$.

4) A sequence $h_1, h_2, \ldots$ of homeomorphisms of E^{n+2} satisfying (i) $h_k \mid E^{n+2} \setminus h_{k-1} \circ \cdots \circ h_1(W^k) = \text{id}$ where $W^k = \bigcup \{W \mid W \in \mathcal{W}_k\}$, (ii) h_k is limited by $h_{k-1} \circ \cdots \circ h_1(\mathcal{W}_k)$, (iii) h_k shrinks $h_{k-1} \circ \cdots \circ h_1(\tilde{G}_k)$ to $1/2^{k+1}$-size, and (iv) $\rho(h_k, \text{id}) < 1/2^{k-1}$ (follows from (ii) and 5).

5) For each $k = 1, 2, \ldots$ and $g \in H_{G_k}$, an open set $U_g \supset g$ so that $\delta(h_k \circ \cdots \circ h_1(U_g)) < 1/2^{k+1}$ and if $j \subset g$ where $j \in H_{G_{k+1}}$, then $U_g \supset W_j \supset U_j$.

For each $k = 1, 2, \ldots$, let $H_k = h_k \circ \cdots \circ h_1$. According to 4)(iv), $\{H_k\}$ forms a Cauchy sequence measured in the sup-norm metric and therefore there is a map $H: E^{n+2} \to E^{n+2}$ with $H_k \to H$. We leave it to the reader to use 1) through 5) to show that $H \mid E^n: E^n \to H(E^n)$ realizes G and therefore E^n/G embeds in E^{n+2} (whenever cl N_G is compact).

If cl N_G is not compact, let $\psi: E^n \to \operatorname{int} I^n \subset E^n$ be a homeomorphism and let $\mathscr{G}$ be the 0-dimensional usc decomposition of E^n induced by ψ and the addition of a single extra nondegenerate element to $\mathscr{G}$, namely ∂I^n. Then

$$E^n/G \xrightarrow{\pi^{-1}} E^n \xrightarrow{\psi} E^n \longrightarrow E^n/\mathscr{G} \hookrightarrow E^{n+2}$$

is an embedding which easily is modified to obtain a closed embedding.

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